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Remarks on Nambu–Poisson and Nambu–Jacobi brackets

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Abstract. We show that Nambu–Poisson and Nambu–Jacobi brackets can be defined inductively: an n-bracket, $n > 2$, is Nambu–Poisson (respectively, Nambu–Jacobi) if and only if on fixing an argument we obtain an $(n-1)$-Nambu–Poisson (respectively, Nambu–Jacobi) bracket. As a by-product we obtain relatively simple proofs of Darboux-type theorems for these structures.

1. Introduction

The concept of a Nambu–Poisson structure was introduced by Takhtajan [Ta] in order to find an axiomatic formalism for the n-bracket operation

$$\{f_1, \ldots, f_n\} = \det\left(\frac{\partial f_i}{\partial x_j}\right) \tag{1.1}$$

proposed by Nambu [Nam] to generalize the Hamiltonian mechanics (cf also [BF, Cha, FDS]). Takhtajan [Ta] has observed that the Nambu canonical bracket (1.1) is n-linear skew-symmetric and satisfies the *generalized Jacobi identity*:

$$\begin{aligned}\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = {} & \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} \\ & +\{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} + \cdots \\ & +\{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}.\end{aligned} \tag{1.2}$$

Such an axiom was also considered by other authors about the same time (see [SV]). These, however, are exactly the axioms of an n-Lie algebra introduced by Filippov [Fi] in 1985, who also gave the example of the canonical Nambu bracket (1.1) in this context. The additional assumption made by Takhtajan was that the bracket, acting on smooth functions, has to satisfy the Leibniz rule

$$\{fg, f_2, \ldots, f_{n-1}\} = f\{g, f_2, \ldots, f_{n-1}\} + \{f, f_2, \ldots, f_{n-1}\}g \tag{1.3}$$

what generalizes the notion of a Poisson bracket and means that the bracket is, in fact, defined by an n-vector field Λ in the obvious way:

$$\{f_1, \ldots, f_n\} = \Lambda_{f_1,\ldots,f_n} \tag{1.4}$$

where we denote $\Lambda_{f_1,\ldots,f_k}$ to be the contraction $i_{\mathrm{d}f_k} \cdots i_{\mathrm{d}f_1}\Lambda$. The generalized Jacobi identity (1.2) means then that the *Hamiltonian vector fields* $\Lambda_{f_1,\ldots,f_{n-1}}$ (of $(n-1)$-tuples of functions this time) preserve the tensor Λ, i.e. the corresponding Lie derivative (which we write as the Schouten bracket) vanishes:

$$[\Lambda_{f_1,\ldots,f_{n-1}}, \Lambda] = 0. \tag{1.5}$$

This also implies that the characteristic distribution D_Λ of the n-vector field Λ, i.e. the distribution generated by all the Hamiltonian vector fields, is involutive. Indeed, from (1.2) we easily derive (for further implications see [DT])

$$[\Lambda_{f_1,\ldots,f_{n-1}}, \Lambda_{g_1,\ldots,g_{n-1}}] = \sum_i \Lambda_{g_1,\ldots,\{f_1,\ldots,f_{n-1},g_i\},\ldots,g_{n-1}}. \tag{1.6}$$

We have even more: the characteristic distribution defines a (possible singular) foliation of the manifold M in the sense of Stefan–Sussmann. This is due to the fact that the module of 1-forms on M is finitely generated over the ring $C^\infty(M)$ of smooth functions on M, so we can generate the distribution by a finite number of Hamiltonian vector fields, which are closed over $C^\infty(M)$ under the Lie bracket and the Stefan–Sussmann condition of integrability of the distribution is satisfied. The proof is exactly parallel to that in the classical Poisson case. All this looks quite similar to the case of classical Poisson structures. Now, the point is that in the case of Nambu–Poisson structures of order $n > 2$ the leaves of the characteristic foliation have to be either 0 or n dimensional; a different behaviour compared with the classical Poisson case. We shall consider this point later.

Since Nambu–Poisson brackets are just the Filippov brackets which are given by multiderivations of the associative algebra $C^\infty(M)$ (the Leibniz rule), the next obvious generalization is to follow the idea of Kirillov [Ki] and to ask what are Filippov n-brackets on the ring of functions, which are given by local operators. They could be called *Nambu–Jacobi* brackets since the Jacobi brackets are what we obtain in the binary case (cf [Gr] for a purely algebraic approach). It is easy to see from (1.2) that every contraction of the bracket with an element f leads from an n-Lie algebra in the sense of Filippov (let us call them simply *n-Filippov algebras*) to an $(n-1)$-Filippov algebra: the $(n-1)$-ary bracket $[\,,\ldots,]_f$ defined by $[f_1,\ldots,f_{n-1}]_f = [f, f_1,\ldots,f_{n-1}]$ satisfies the $(n-1)$-Jacobi identity if the bracket $[\,,\ldots]$ satisfies the n-Jacobi identity. Hence, contractions lead from n-Nambu–Jacobi brackets to $(n-1)$-Nambu–Jacobi brackets (we will show later that it can be converted). The binary Jacobi brackets are given by first-order differential operators [Ki, Gr], so, due to the fact that they are totally skew-symmetric, n-Nambu–Jacobi brackets are also given by first-order differential operators. Similarly, as in the binary case, they can be written with the help of two multivector fields: the n-vector field Δ and the $(n-1)$-vector field Γ, in the form

$$\{f_1,\ldots,f_n\} = (\Delta + s(\Gamma))(f_1,\ldots,f_n) \tag{1.7}$$

where $\Delta(f_1,\ldots,f_n) = \Delta_{f_1,\ldots,f_n}$ is just the bracket induced by Δ and

$$s(\Gamma)(f_1,\ldots,f_n) = \sum_i (-1)^{i+1} f_i \Gamma_{f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_n} \tag{1.8}$$

(cf [MVV]). The n-Jacobi identity puts additional restrictions on the pair (Δ,Γ) to obtain a Nambu–Jacobi structure. For the classical case, $n = 2$, they read

$$[\Gamma,\Delta] = 0 \tag{1.9}$$

$$[\Delta,\Delta] = -2\Gamma\wedge\Delta \tag{1.10}$$

where the brackets are the Schouten brackets. We use the Schouten bracket

$$\begin{aligned}[X_1\wedge\ldots\wedge X_k, Y_1\wedge\ldots\wedge Y_k] = \sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\\ \ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_k\wedge Y_1\wedge\ldots\wedge\widehat{Y}_j\wedge\ldots\wedge Y_l.\end{aligned} \tag{1.11}$$

Let us note that sometimes one can meet a version of (1.10) differing in sign (cf [Li, DLM]) when the bracket differs in sign from the bracket we use.

In this paper we prove the inductive property of the Nambu–Poisson and Nambu–Jacobi bracket: for $n > 2$, an n-linear skew-symmetric bracket of smooth functions on a manifold is a Nambu–Poisson (respectively, Nambu–Jacobi) bracket if and only if on fixing one argument we obtain a Nambu–Poisson (respectively, Nambu–Jacobi) bracket of order $(n-1)$. As a by-product we obtain versions of Darboux-type theorems for these bracket (cf [AG, Ga, Na1, Pa, MVV]) with relatively short proofs.

There are other concepts of n-ary Lie, Poisson and Jacobi brackets using a generalized Jacobi identity of different type than (1.2), a skew-symmetrization of it. We will not discuss them here, so let us only mention the papers [APP, AIP, ILMD, ILMP, MV] and references therein. Recently, a unification of different concepts was proposed in [VV].

2. Recursive characterization of Nambu–Poisson and Nambu–Jacobi algebras

We start with the following easy observation.

Proposition 1. *Let $X_1, \ldots, X_n$ be vector fields on a manifold M. Then $\Lambda = X_1 \wedge \ldots \wedge X_n$ is a Nambu–Poisson tensor if and only if the distribution D generated by these vector fields is involutive at regular points of Λ.*

Proof. The proposition has a local character, so all considerations will be local near regular points and we may assume that the vector fields are linearly independent. Under this condition D coincides with the characteristic distribution D_Λ of the tensor field Λ, i.e.

$$D = \mathrm{span}\{\Lambda_{f_1,\ldots,f_{n-1}} : f_i \in C^\infty(M), i = 1, \ldots, n-1\}. \tag{2.1}$$

If Λ is a Nambu–Poisson tensor then $D_\Lambda = D$ is known to be involutive. On the other hand, if D is involutive, then it generates an n-dimensional foliation which, in appropriate coordinates, is generated by the coordinate vector fields $\partial_1, \ldots, \partial_n$ (a version of the Frobenius theorem), so that $\Lambda = f\partial_1 \wedge \ldots \wedge \partial_n$ with some function f. This is a standard example of a Nambu–Poisson tensor (cf [MVV], corollary 3.2). As a matter of fact, the function f could be chosen to be a constant. □

Remark. At singular points of Λ the situation may be different. For instance, $\Lambda = \partial_1 \wedge (x_1\partial_2)$ is a Poisson tensor, but D is not involutive at points, where $x_1 = 0$: $[\partial_1, x_1\partial_2] = \partial_2 \notin D$.

We shall make use of the following variant of the lemma 'on three planes' (cf [MVV] or [DZ]).

Lemma 1. *Let $\{\Lambda_i : i \in I\}$ be a family of decomposable non-zero n-vectors of a vector space V such that every sum $\Lambda_{i_1} + \Lambda_{i_2}$ is again decomposable. Then,*

(a) the linear span D of the linear subspaces D_{Λ_i} they generate is at most $(n+1)$-dimensional or

(b) the intersection $\cap_i D_{\Lambda_i}$ is at least $(n-1)$ dimensional.

Proof. It is easy to see that the sum $\Lambda_{i_1} + \Lambda_{i_2}$ is decomposable, where the summands are non-zero, if and only if the intersection of n-dimensional subspaces $D_{\Lambda_{i_1}} \cap D_{\Lambda_{i_2}}$ is at least $(n-1)$ dimensional. Then we can use a corrected version of the 'lemma on three planes' ([MVV], lemma 4.4) as in [DZ], which states that in this case we have (a) or (b), with a rather obvious proof. □

Lemma 2. *Let Λ be an n-vector field on a manifold M, $n > 2$, such that all the contractions Λ_f, with $f \in C^\infty(M)$, are Nambu–Poisson tensors. Then Λ is decomposable at its regular points.*

Proof. The Nambu–Jacobi identity for Λ_f reads

$$[\Lambda_{f,f_1,\ldots,f_{n-2}}, \Lambda_f] = 0 \tag{2.2}$$

where [,] stands for the Schouten bracket. It easily implies that the operation A acting on functions by

$$A(f_1, \ldots, f_{n-1}, g_1, \ldots, g_n) = [\Lambda_{f_1,\ldots,f_{n-1}}, \Lambda]_{g_1,\ldots,g_n} \tag{2.3}$$

is totally skew-symmetric (being skew-symmetric with respect to f_i's, g_i's and vanishing for $f_1 = g_1$) and it is represented by a $(2n-1)$-vector field, since it acts by derivation on g_i's. This implies that

$$A(f_1^2, f_2, \ldots, f_{n-1}, g_1, \ldots, g_n) = 2f_1 A(f_1, f_2, \ldots, f_{n-1}, g_1, \ldots, g_n). \tag{2.4}$$

On the other hand, from properties of the Schouten bracket we obtain

$$\begin{aligned} A(f_1^2, f_2, \ldots, f_{n-1}, g_1, \ldots, g_n) &= [2f_1\Lambda_{f_1,f_2,\ldots,f_{n-1}}, \Lambda]_{g_1,\ldots,g_n} \\ &= 2f_1 A(f_1, f_2, \ldots, f_{n-1}, g_1, \ldots, g_n) \pm 2(\Lambda_{f_1,f_2,\ldots,f_{n-1}} \wedge \Lambda_{f_1})_{g_1,\ldots,g_n} \end{aligned} \tag{2.5}$$

so that

$$\Lambda_{f_1,f_2,\ldots,f_{n-1}} \wedge \Lambda_{f_1} = 0. \tag{2.6}$$

The last identity, for $n > 2$, implies that Λ is decomposable at regular points, as shown in [MVV], proposition 4.1, or [Ga]. □

Theorem 1. *An n-vector field Λ, $n > 2$, on a manifold M is a Nambu–Poisson tensor if and only if its contractions Λ_f are Nambu–Poisson tensors for all $f \in C^\infty(M)$. In this case, the tensor Λ can be written in appropriate coordinates around its regular points in the form $\partial_1 \wedge \ldots \wedge \partial_n$.*

Proof. The implication $\Rightarrow$ is well known and trivial. To show the converse, we can first reduce to regular points and then make use of lemma 2 to find that Λ is decomposable, say $\Lambda = X_1 \wedge \ldots \wedge X_n$. Now, we have to show that the distribution D generated by the linearly independent vector fields $X_1, \ldots, X_n$ is involutive. Since X_1 is (locally) non-vanishing, we can find a function f such that (again locally) $X_1(f) \equiv 1$. Putting now $X_i' = X_i - X_i(f)X_1$ for $i > 1$, we obtain $\Lambda = X_1 \wedge X_2' \wedge \ldots \wedge X_n'$ with $X_i'(f) = 0$. Thus $\Lambda_f = X_2' \wedge \ldots \wedge X_n'$ is a Nambu–Poisson tensor by the inductive assumption, so that, in certain local coordinates, $\Lambda_f = \partial_1 \wedge \ldots \wedge \partial_{n-1}$ as in the proof of proposition 1, and

$$\Lambda = X_1 \wedge \Lambda_f = X_1 \wedge \partial_1 \wedge \ldots \wedge \partial_{n-1}. \tag{2.7}$$

In fact, we can additionally assume that $X_1(x_i) = 0$ for $i = 1, \ldots, n-1$, replacing X_1 by $X_1 - \sum_1^{n-1} X_1(x_i)\partial_i$. Assuming that the characteristic distribution D generated by Λ is not involutive, we would find that $[\partial_i, X_1] \notin D$ for some i, say $i = n-1$. However, then $\Lambda_{x_1} = -X_1 \wedge \partial_2 \wedge \ldots \wedge \partial_{n-1}$ generates a distribution, which is not involutive, contrary to the inductive assumption. □

Corollary 1. *An n-linear skew-symmetric bracket $\{,\ldots,\}$ on functions on a manifold M, $n > 2$, is a Nambu–Poisson bracket if and only if its contraction with any $(n-2)$ functions gives a Poisson bracket.*

Proof. Since the contractions give a Poisson bracket, our bracket operation is given by an n-linear first-order differential operator vanishing on constants, so by an n-vector field Λ. The rest follows by applying theorem 1 recursively. □

We have a similar theorem for Nambu–Jacobi brackets. They have to be of the form $\Delta + s(\Gamma)$ (cf [MVV]) and we obtain particular cases: just the Nambu–Poisson structure Δ (locally $\Gamma = 0$) and the bracket given by $s(\Gamma)$ (locally, $\Delta = 0$). We find other examples by putting in local coordinates $\Delta = \partial_1 \wedge \ldots \wedge \partial_n$ and $\Gamma = \partial_1 \wedge \ldots \wedge \partial_{n-1}$. The characterization theorem [MVV] shows that this is a rather general picture. The following inductive theorem will suggest an alternative proof of this characterization theorem.

Theorem 2. *An n-linear skew-symmetric bracket $\{,\ldots,\}$ on functions on a manifold M, $n > 2$, is a Nambu–Jacobi bracket if and only if its contraction $\{f, ,\ldots,\}$ with any function f is a Nambu–Jacobi bracket. Moreover,*

$$\{f_1,\ldots,f_n\} = \Delta_{f_1,\ldots,f_n} + s(\Gamma)(f_1,\ldots,f_n) \tag{2.8}$$

for some multivector fields Δ and Γ which, in some local coordinates around points where they do not vanish, can be written in the form $\Delta = \partial_1 \wedge \ldots \wedge \partial_n$ and $\Gamma = \partial_1 \wedge \ldots \wedge \partial_{n-1}$. If, locally, one of the tensors Δ, Γ vanishes, the other is a Nambu–Poisson tensor (Γ is an ordinary Poisson tensor of rank 2 if $n = 3$) and can be written as above around its regular points.

Proof. Only the implication $\Leftarrow$ is non-trivial. Since the contractions are given by first-order differential operators, our bracket is given by an n-linear first-order differential operator, i.e. (cf equation (1.7))

$$\{f_1,\ldots,f_n\} = \Delta_{f_1,\ldots,f_n} + s(\Gamma)(f_1,\ldots,f_n) \tag{2.9}$$

for an n-vector field Δ and an $(n-1)$-vector field Γ. It is easy to see that the analogous decomposition for the contraction with the function f yields

$$\{f,\cdot,\ldots,\cdot\} = (\Delta_f + f\Gamma) - s(\Gamma_f) \tag{2.10}$$

i.e. $(\Delta_f + f\Gamma, -\Gamma_f)$ is a Nambu–Jacobi structure for all $f \in C^\infty(M)$. In particular, for $f \equiv 1$, we find that $(\Gamma, 0)$ is a Nambu–Jacobi structure, which implies that Γ is a Nambu–Poisson tensor. If, locally, $\Delta \equiv 0$, then our original bracket is of the form $s(\Gamma)$, so it is a Nambu–Jacobi structure. If, on the other hand, Γ vanishes locally, then our original bracket is just given by Δ. However, the contractions Δ_f give Nambu–Jacobi and hence Nambu–Poisson ($\Gamma = 0$) structures, so our bracket is a Nambu–Poisson bracket in view of theorem 1.

Since our theorem is local and it suffices to check the Jacobi identities on an open-dense subset, we may now assume that $\Delta \neq 0$ and $\Gamma \neq 0$.

2.1. The case $n = 3$

Since for all $f \in C^\infty(M)$ the pairs $(\Delta_f + f\Gamma, -\Gamma_f)$ constitute the usual Jacobi structures, according to (1.10), we have the identity

$$[\Delta_f + f\Gamma, \Delta_f + f\Gamma] = 2\Gamma_f \wedge (\Delta_f + f\Gamma). \tag{2.11}$$

Computing the Schouten brackets and using the fact that Γ is an ordinary Poisson structure ($[\Gamma, \Gamma] = 0$ and $[\Gamma_f, \Gamma] = 0$), we obtain

$$[\Delta_f, \Delta_f] + 2f[\Delta_f, \Gamma] - 2f\Gamma_f \wedge \Gamma = 2\Gamma_f \wedge \Delta_f + 2f\Gamma_f \wedge \Gamma \tag{2.12}$$

(let us notice that $[\Gamma, f] = -\Gamma_f$) which gives

$$[\Delta_f, \Delta_f] + 2f[\Delta_f, \Gamma] - 2\Gamma_f \wedge \Delta_f - 4f\Gamma_f \wedge \Gamma = 0. \tag{2.13}$$

Putting $f := f + 1$ in (2.13), we see that

$$[\Delta_f, \Gamma] - 2\Gamma_f \wedge \Delta = 0 \tag{2.14}$$

and hence

$$[\Delta_f, \Delta_f] + 2\Gamma_f \wedge \Delta_f = 0. \tag{2.15}$$

Further, replacing f by f^2 in (2.14), we obtain

$$\begin{aligned} 0 &= [\Delta_{f^2}, \Gamma] - 2\Gamma_{f^2} \wedge \Delta = [2f\Delta_f, \Gamma] - 4f\Gamma_f \wedge \Delta \\ &= 2f[\Delta_f, \Gamma] - 2\Gamma_f \wedge \Delta_f - 4f\Gamma_f \wedge \Delta = -2\Gamma_f \wedge \Delta_f \end{aligned} \tag{2.16}$$

which, compared with (2.15), gives

$$[\Delta_f, \Delta_f] = 0 \tag{2.17}$$

for all $f \in C^\infty(M)$. The latter equation means that Δ_f is a Poisson tensor for each $f \in C^\infty(M)$, so Δ is itself a Nambu–Poisson tensor (and hence decomposable) according to theorem 1.

Assume now that $\Delta \neq 0$. In view of (2.16), Γ_f divides the decomposable tensor Δ_f, if only $\Delta_f \neq 0$, and hence also divides Δ. If $\Delta_f = 0$ at a given point, we can use the linearization

$$\Gamma_f \wedge \Delta_g + \Gamma_g \wedge \Delta_f = 0 \tag{2.18}$$

of (2.16) to obtain (at this point) $\Gamma_f \wedge \Delta_g = 0$, for g chosen such that $\Delta_g \neq 0$, and to conclude that $\Gamma_f \wedge \Delta = 0$ for all f.

Now, similarly as in ([MVV], theorem 5.1), we can find local coordinates such that $\Gamma = \partial_1 \wedge \partial_2$ and $\Delta = \phi\partial_1 \wedge \partial_2 \wedge \partial_3$ for some function ϕ. Now, since (2.13) implies that $[\Gamma_f, \Delta_f] = 0$, and putting $f = x_i$, $i = 1, 2$, we find that $[\partial_i, \phi\partial_i \wedge \partial_3] = 0$, $i = 1, 2$, so that the vector fields $\partial_1, \partial_2, \phi\partial_3$ pairwise commute and we can chose local coordinates so that $\phi \equiv 1$. In particular, $\Gamma = \Delta_{x_3}$.

If, locally, $\Delta = 0$, then (2.13) reduces to $f\Gamma_f \wedge \Gamma = 0$ for all f and hence Γ is decomposable, i.e. Γ is an ordinary Poisson tensor of rank 2.

2.2. The case $n > 3$

Since $(\Delta_f + f\Gamma, -\Gamma_f)$ is a Nambu–Jacobi structure of order $(n-1) > 2$, $\Delta_f + f\Gamma$ is a decomposable (at its regular points) Nambu–Poisson tensor. Let D_f be its characteristic distribution. The decomposable tensor $\Delta_{f+g} + (f+g)\Gamma$ is the sum of two decomposable tensors $(\Delta_f + f\Gamma) + (\Delta_g + g\Gamma)$ at their regular points, so the dimension of $D_f \cap D_g$ is at least $(n-2)$ if $D_f, D_g \neq \{0\}$. It is easy to see that the intersection of non-trivial D_f must be zero.

Indeed, if a vector field X divides all $\Delta_f + f\Gamma$, then X divides Γ (put $f \equiv 1$) and hence all Δ_f, so that $\Delta_f = X \wedge Y^f$, where Y^f is an $(n-2)$-vector field (all is local and for 'most' functions f we have $\Delta_f \neq 0$). Finding a function g such that $X(g) = 1$, we can assume, as in the proof of theorem 1, that $Y_g^f = 0$, so that $\Delta_{f,g} = Y^f \neq 0$. Similarly, $\Delta_{g,g} = Y^g$, but $\Delta_{g,g} = 0$, so $\Delta_g = X \wedge Y^g = 0$; a contradiction, since $\Delta_{f,g} = -\Delta_{g,f} \neq 0$. Therefore, lemma 1 implies that the linear span D of all the distributions D_f has the dimension $\leqslant n$. Moreover, since $D_\Gamma = D_1$, also $D_\Gamma \subset D$. Now,

$$(\Delta_{f_1} + f_1\Gamma)_{f_2,\ldots,f_{n-1}} = \Delta_{f_1,f_2,\ldots,f_{n-1}} + f_1\Gamma_{f_2,\ldots,f_{n-1}} \in D \tag{2.19}$$

implies that $\Delta_{f_1,f_2,\ldots,f_{n-1}} \in D$, i.e. the characteristic distribution D_Δ of Δ is contained in D. However, Δ is an n-tensor, so $\dim(D_\Delta) \geqslant n$ at regular points and hence $D_\Delta = D$ and Δ is decomposable at its regular points.

This implies that we can find local coordinates such that $\Gamma = \partial_1 \wedge \ldots \wedge \partial_{n-1}$ and $\Delta = \partial_1 \wedge \ldots \wedge \partial_{n-1} \wedge X$ for a vector field X, which we may assume to annihilate the coordinate functions $x_1, \ldots, x_{n-1}$. If the characteristic distribution $D = D_\Delta$ are not involutive, say $[\partial_{n-1}, X] \notin D$ then the Nambu–Poisson tensor

$$\Delta_{x_1} + x_1\Gamma = \partial_2 \wedge \ldots \wedge \partial_{n-1} \wedge (X \pm x_1\partial_1) \tag{2.20}$$

would have a non-involutive characteristic distribution; a contradiction. Therefore, Δ is a Nambu–Poisson tensor and we can write the vector field X in the form $X = \phi\partial_n$ with a function ϕ. Let us now show that the Hamiltonian vector fields of Γ preserve Δ. Indeed, since $(\Delta_f + f\Gamma, -\Gamma_f)$ is a Nambu–Jacobi structure, at regular points $-\Gamma_f = (\Delta_f + f\Gamma)_h$ for some function h and hence

$$-[\Gamma_{f,f_1,\ldots,f_{n-3}}, \Delta_f + f\Gamma] = [(\Delta_f + f\Gamma)_{h,f_1,\ldots,f_{n-3}}, \Delta_f + f\Gamma] = 0 \tag{2.21}$$

$(\Delta_f + f\Gamma$ is a Nambu–Poisson tensor). However, $[\Gamma_{f,f_1,\ldots,f_{n-3}}, f\Gamma] = 0$ (Γ is a Nambu–Poisson tensor), so that

$$[\Gamma_{f,f_1,\ldots,f_{n-3}}, \Delta_f] = 0. \tag{2.22}$$

Now putting $(f, f_1, \ldots, f_{n-3}) = (x_i, x_1, \ldots, \check{x}_i, \ldots, \check{x}_j, \ldots, x_{n-1})$, where $\check{x}_i$ stands for omission, we find, similarly as in the case $n = 3$, that $\partial_1, \ldots, \partial_{n-1}, \phi\partial_n$ are pairwise commuting vector fields, so we can put $\phi \equiv 1$ in appropriate local coordinates. In particular, $\Gamma = (-1)^n\Delta_{x_n}$. $\square$

3. Conclusions

Over the last few years interest in n-ary generalizations of the concept of Lie algebra, especially in Nambu–Poisson and Nambu–Jacobi brackets, has been growing among mathematicians and physicists.

We have proved that the Nambu–Poisson and the Nambu–Jacobi bracket can be defined inductively: a bracket is such an n-bracket if and only if, when contracted with any function, it gives such an $(n-1)$-bracket. This reduces the general case to classical Poisson and Jacobi structures. In the proof we have used the fact that these brackets are given by decomposable multivector fields defining first-order differential operators on the algebra of smooth functions on a manifold.

The question, of whether an analogue of this fact is true for any (finite-dimensional) Filippov algebra, remains open. Any Filippov n-algebra structure on a vector space V is defined by a linear multivector field

$$\Lambda = \sum_{i_1,\dots,i_n} c^k_{i_1,\dots,i_n} x_k \partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_n}} \tag{3.1}$$

on V^*, where a basis (x_i) of V is regarded as a coordinate system for V^*. This tensor field defines an n-bracket on linear functions, which should satisfy the generalized Jacobi identity. In general, however, such tensors need not be decomposable, since direct products of the Filippov algebras correspond to 'direct sums' of the corresponding tensors, which can never be decomposable (if the summands are non-zero). Therefore, we finish with the following problem.

Problem. *Let* $[\cdot,\dots,\cdot]$ *be an* n*-bracket,* $n>2$*, on a finite-dimensional vector space* V *such that* $[y_1,\dots,y_{n-1}]_x = [x, y_1,\dots,y_{n-1}]$ *is a Filippov* $(n-1)$*-bracket for every* $x \in V$*. Does it imply that* $[\cdot,\dots,\cdot]$ *is Filippov itself?*

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